

Generic model of an atom laser

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(Submitted to Physical Review A: June 18, 1998; revised manuscript submitted: August 20, 1998)

We present a generic model of an atom laser by including a pump and loss term in the Gross-Pitaevskii equation. We show that there exists a threshold for the pump above which the mean matter field assumes a non-vanishing value in steady-state. We study the transient regime of this atom laser and find oscillations around the stationary solution even in the presence of a loss term. These oscillations are damped away when we introduce a position dependent loss term. For this case we present a modified Thomas-Fermi solution that takes into account the pump and loss. Our generic model of an atom laser is analogous to the semi-classical theory of the laser.

PACS number(s): 03.75.Fi, 05.30.Jp, 42.55.Ah

I. INTRODUCTION

With the recent experiments on Bose-Einstein condensation [1–13] an atom laser, that is a device which produces an intense coherent beam of atoms by a stimulated process [14,15], has become feasible. Already, the MIT group has realized a pulsed atom laser [16] and has provided evidence for the process of coherent matter-wave amplification in the formation of a Bose condensate [17].

How to describe the matter field of an atom laser? Rate equations are simple. But they cannot answer this question since they do not contain any coherence. In contrast a microscopic and fully quantum mechanical treatment can answer the question but is not easy to handle. What we need is a theory that includes coherence but is still simple. In the present paper we develop such a theory—a generic model of the atom laser.

The optical laser has three essential ingredients: (i) a resonator for the electromagnetic field, (ii) an atomic medium, and (iii) an excitation mechanism for the atoms. We start the laser cycle by preparing the atoms in an excited state using this excitation mechanism, which in general is incoherent. The radiation emitted by the atom amplifies the electromagnetic field in a mode of the resonator. To be efficient the frequency of the mode has to match appropriately the frequency of the transition. The boundary conditions set by the resonator determine via the Helmholtz equation the spatial part of the mode function. The ultimate goal is to transfer the excitation of the atom into a macroscopic excitation of the field mode. In this way we transfer the energy used to excite the atom via the gain medium into coherent excitation of the field mode. In order to make use of the radiation we have to couple it out of the resonator. We compensate for this loss by continuously re-pumping the medium.

The goal of an atom laser is completely analogous: We want to create a macroscopic coherent excitation of a mode in a resonator for atoms. Hence, in an atom laser the atoms play the role of the field excitation of the optical laser. Since atoms cannot be created or annihilated

the means to achieve lasing are different. Indeed, in an atom laser there is no “real” laser medium: It is the same atom that goes through the laser process. We only manipulate the internal degrees of freedom and center-of-mass motion of the atoms. In particular we want to force their center-of-mass motion into a specific quantum state of the resonator. The resonator for the atoms is a binding potential such as provided by a trap. The spatial part of the mode function of this atomic resonator follows from the time independent Schrödinger equation [18]. Moreover, we focus on the ground state of the trap. As in the optical laser we want a macroscopic excitation of this mode, that is we strive to have as many atoms as possible in one quantum state. This is the phenomenon of Bose-Einstein condensation. As in the optical laser we need to couple the atomic wave out of the atomic resonator. In order to have a continuous wave (cw) atom laser, we have to continuously feed in more atoms.

There exist two different approaches towards a theoretical description of a cw atom laser: The first one relies on rate equations [19–21] whereas the second one derives a quantum mechanical master equation [22–27]. In the present paper we suggest a third approach which makes heavily use of the close analogy between an atom laser and an optical laser. In the latter case it turned out that a classical treatment of the electromagnetic field [28,29] was sufficient to describe many features of the laser. Can we therefore devise a semi-classical theory of the atom laser?

The semi-classical laser theory replaces the electromagnetic field operator $\hat{E}(\mathbf{r}, t)$ for the field inside the laser cavity by the expectation value $\mathcal{E}(\mathbf{r}, t) \equiv \langle \hat{E}(\mathbf{r}, t) \rangle$. The equation of motion for \mathcal{E} is the wave equation of Maxwell’s electrodynamics driven by the polarization of the laser medium. An additional term introduced phenomenologically takes into account the loss of the cavity. The polarization of the laser medium follows from a microscopic, quantum mechanical description of the internal structure of the atoms. Quantum mechanics rules the atoms whereas classical Maxwell’s wave theory determines the electromagnetic field. These are the essential

ideas of semi-classical laser theory.

The semi-classical laser equations do not prefer any particular phase. Nevertheless by choosing an arbitrary phase we can describe many properties of the electromagnetic field. Furthermore, we have to start with a non-vanishing seed field in order to obtain a non-vanishing solution for the electromagnetic field with a fixed phase.

In our model of an atom laser we replace the matter-wave field represented by a field operator $\hat{\Psi}(\mathbf{r}, t)$ by a scalar mean field $\psi(\mathbf{r}, t) \equiv \langle \hat{\Psi}(\mathbf{r}, t) \rangle$. This is analogous to replacing the field operator \hat{E} by its expectation value \mathcal{E} in semi-classical laser theory. The well-known Gross-Pitaevskii equation [30,31] plays now the role of Maxwell's wave equation. It defines the equation of motion for $\psi(\mathbf{r}, t)$. Similar to the semi-classical theory of the optical laser we have to break the symmetry of the equation of motion for ψ in order to have a non-vanishing value for ψ . For massive particles this is more problematic than for photons since they cannot be created or annihilated, and for all quantum states with fixed particle number we have $\langle \hat{\Psi} \rangle = 0$. Nevertheless the concept of spontaneously broken symmetry turned out to be very useful to describe properties of a condensate, in particular interference effects. For more detailed discussions see Refs. [15,32]. In contrast to the driven electromagnetic wave equation the Gross-Pitaevskii equation does not contain a gain term analogous to the polarization. This reflects the fact that in Bose-Einstein condensation there is no "medium" in the trap. We therefore add a phenomenological pump term. Moreover, as in the electromagnetic case we have to add a loss term.

Despite the similarity there is a fundamental difference in the two equations of motion. In the absence of a medium Maxwell's wave equation is linear. In contrast the Gross-Pitaevskii equation is non-linear. This is a manifestation of the interaction of atoms.

The crucial part of any laser is the stimulated amplification process. Different mechanisms for matter-wave amplification have been suggested and discussed: optical cooling [19–23], elastic collisions by evaporative cooling [25,26,33], dissociation of molecules [27], and cooling by a thermal reservoir [34]. For different schemes of pumping a condensate we refer to Refs. [35–37]. Our model does not rely on a specific mechanism, but can be adapted to any mechanism where Bose enhancement is present.

The paper is organized as follows: In Sec. II, we generalize the Gross-Pitaevskii equation by including gain and loss terms. One important loss is due to coupling the atom wave out of the resonator. However, similar to semi-classical laser theory where usually the field inside the laser cavity is investigated, we restrict ourselves to the matter-wave field inside the resonator. We therefore do not go into details of an output coupler [38–43]. We present the stationary solutions of our equations and perform a stability analysis. We find a threshold behavior similar to the optical laser. Moreover, we calculate the time dependent solution and show how it converges

to a quasi-stationary solution. In general, the quasi-stationary solution we find does not coincide with the time independent solution but shows some oscillatory behavior around it. These oscillations disappear when we modify our equations in Sec. III by introducing a space dependent loss. This loss can be thought of as a consequence of collisions between condensed and un-condensed atoms at the edges of the condensate. For this improved model we present a modified Thomas-Fermi approximation for the stationary solution. Section IV summarizes our results.

II. ELEMENTARY MODEL

In this section we summarize our generic model of an atom laser.

A. Formulation of the model

In particular, we add the phenomenological gain and loss terms to the Gross-Pitaevskii equation (GPE). In this way we couple the GPE to an equation governing the number of un-condensed atoms.

The Gross-Pitaevskii equation (GPE) for the mean field $\psi = \psi(\mathbf{r})$ of the condensed atoms of mass m in a trap potential $V(\mathbf{r})$ reads [44,45]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\mathbf{r})\psi + U_0 |\psi|^2 \psi. \quad (1)$$

The non-linear term $U_0 |\psi|^2 \psi$ with $U_0 = 4\pi\hbar^2 a_s / m$ takes into account two-particle interactions where a_s denotes the s-wave scattering length.

The GPE has been very successful in describing the properties of Bose-Einstein condensates. However, in the present form it cannot describe the growth of or the loss of atoms out of a condensate. Indeed, the GPE keeps the number of atoms

$$N_c \equiv \int |\psi(\mathbf{r})|^2 d^3r \quad (2)$$

constant.

In order to overcome this problem we introduce two additional terms in the GPE: a loss term and a gain term. The loss term

$$H_{\text{loss}} \psi \equiv -\frac{i\hbar}{2} \gamma_c \psi \quad (3)$$

leads to an exponential decay of the number of atoms in the condensate. In contrast the gain term

$$H_{\text{gain}} \psi \equiv \frac{i\hbar}{2} \Gamma N_u \psi \quad (4)$$

leads to an increase of atoms in the condensate. Here N_u is the number of atoms outside the condensate, that is,

the un-condensed atoms, and Γ is the rate for the transition of these atoms into the condensate. We regard this as a generic pump mechanism [46] of an atom laser since it contains already the Bose enhancement as we will see later.

When we add the loss term (3) and the gain term (4) to the GPE (1), we arrive at the generalized GPE

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\mathbf{r})\psi + U_0 |\psi|^2 \psi - \frac{i}{2} \hbar \gamma_c \psi + \frac{i}{2} \hbar \Gamma N_u \psi. \quad (5)$$

For the number of un-condensed atoms N_u we assume the rate equation

$$\dot{N}_u = R_u - \gamma_u N_u - \Gamma N_c N_u. \quad (6)$$

The first term reflects the fact that we generate with a rate R_u atoms in an un-condensed state from an infinite reservoir of atoms. The term $-\gamma_u N_u$ takes into account that atoms can escape from our system without being trapped in the condensed state. The last term describes the transition of the atoms into the condensate. Since it is proportional to the number of atoms N_c in the condensate, it contains the Bose-enhancement factor.

In this way we have coupled the generalized GPE governing the condensate to the rate equation governing the number of un-condensed atoms. These two equations (5) and (6) are the foundations of our model.

The number N_c of condensed atoms follows from the generalized GPE (5) with the help of the definition, Eq. (2). From Eq. (5) we obtain the rate equation

$$\dot{N}_c = \Gamma N_u N_c - \gamma_c N_c \quad (7)$$

for the number of atoms in the condensate.

We conclude this section by noting that rate equations similar to Eqs. (6) and (7) have already been discussed in the literature [19,20]. However, in the present paper we replace the rate equation (7) by the generalized GPE (5). This equation obviously contains the rate equation but in addition the coherence.

B. Stationary solutions and stability analysis

Before we turn to the stationary solution of the generalized GPE, Eq. (5), we first discuss the stationary solutions of the rate equations (6) and (7).

1. Rate equations

The rate equations (6) and (7) have two possible stationary solutions:

The solution

$$N_c^{(s)} \equiv 0, \quad N_u^{(s)} = R_u / \gamma_u \quad (8)$$

is reminiscent of the optical laser below threshold where the intensity of the laser vanishes.

The other stationary solution

$$N_c^{(s)} = \frac{R_u}{\gamma_c} - \frac{\gamma_u}{\Gamma}, \quad N_u^{(s)} = \frac{\gamma_c}{\Gamma} \quad (9)$$

corresponds to the laser above threshold.

We now perform a stability analysis of these solutions. For this purpose we introduce small deviations

$$n_u(t) \equiv N_u(t) - N_u^{(s)}, \quad n_c(t) \equiv N_c(t) - N_c^{(s)}, \quad (10)$$

and arrive at the linearized equations

$$\dot{n}_u = -(\gamma_u + \Gamma N_c^{(s)})n_u - \Gamma N_u^{(s)}n_c, \quad \dot{n}_c = -(\gamma_c - \Gamma N_u^{(s)})n_c + \Gamma N_c^{(s)}n_u. \quad (11)$$

A stability analysis of these equations shows that the stationary solution in Eqs. (8) is stable for

$$R_u < R^{th} \equiv \frac{\gamma_c \gamma_u}{\Gamma}. \quad (12)$$

Likewise, our stability analysis shows that Eqs. (9) are a stable stationary solution for

$$R_u > \frac{\gamma_c \gamma_u}{\Gamma} = R^{th}. \quad (13)$$

Therefore, there exists a threshold R^{th} . When the pump rate R_u is below the threshold, the number N_c of atoms in the condensate vanishes. When it is above, it is non-vanishing. In that case the steady-state number of atoms in the condensate grows linearly with the pump rate R_u .

2. Generalized GPE

We now turn to the discussion of the stationary solutions of our generic model of the atom laser, that is of the generalized GPE coupled to the rate equation for N_u .

The stable stationary solution of Eq. (5) corresponding to a vanishing number of atoms in the condensate is

$$\psi^{(s)} \equiv 0. \quad (14)$$

This is the stationary solution of the mean field below threshold.

Above threshold the stationary solution $\psi^{(s)}$ of the generalized GPE (5) is identical to the stationary solution of the conventional GPE. Indeed, when we substitute the ansatz $\psi = \exp(-i\mu t/\hbar)\psi^{(s)}$ into Eq. (5) and note that the gain and loss terms cancel each other as a result of the stationary solution, Eq. (9), we arrive at

$$\mu\psi^{(s)} = -\frac{\hbar^2}{2m}\Delta\psi^{(s)} + V(\mathbf{r})\psi^{(s)} + U_0|\psi^{(s)}|^2\psi^{(s)}. \quad (15)$$

Here, μ denotes the chemical potential.

A stability analysis of this equation leads to collective excitations [47] which in general cannot be treated analytically. For such an analysis in the case of a one-dimensional harmonic oscillator we refer to the Appendix.

C. Transient behavior

In the preceding section we have discussed the steady-state solutions of both the rate equations (6) and (7) and the matter-field equations (5) and (6) of the atom laser. In the present section we address the question if and how the matter field evolves into a stationary state from an initial condition. Here we focus on the case above threshold. Since the equations are non-linear we solve them numerically for the case of an one-dimensional harmonic oscillator potential

$$V(x) = \frac{1}{2}m\omega^2x^2. \quad (16)$$

This one-dimensional trap potential already shows the essential features of the atom laser.

We therefore analyze the one-dimensional generalized GPE

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2 + U_x|\psi(x,t)|^2 - \frac{i}{2}\hbar\gamma_c + \frac{i}{2}\hbar\Gamma N_u \right] \psi(x,t), \quad (17)$$

describing the matter field, coupled to the equation

$$\dot{N}_u = R_u - \gamma_u N_u - \Gamma N_c N_u \quad (18)$$

for the number of un-condensed atoms. We use the split-operator method [48,49] to solve the generalized GPE. This technique has already been used successfully for the ordinary GPE [50]. Moreover, we supplement this numerical analysis by an analytical treatment of the solution in the long-time limit where we apply the method of Ref. [51] to one dimension. For the details of the analytical approach we refer to the Appendix.

1. Initial conditions and parameters

The semi-classical theory of the optical laser cannot explain the initial start-up of the laser since it does not contain spontaneous emission: A non-vanishing seed electromagnetic field starts the laser. Similarly, in our model of the atom laser we start from an initial condition for the mean matter field $\psi(x, t=0)$ that is non-zero. This is the seed field for our atom laser.

Let us illustrate this by considering for the moment the “natural” initial condition $\psi(x, t=0) = 0$ and $N_u(0) = 0$. This implies $N_c(0) = 0$. We therefore start without any atoms in the system. Moreover, we consider a pump rate R_u above threshold. This choice of initial conditions leads to the unstable solution $N_c = 0$ and $N_u = R_u/\gamma_u$. Indeed, any small perturbation in N_c leads to a completely different behavior and N_c and N_u approach the stable solutions $N_c^{(s)} = R_u/\gamma_c - \gamma_u/\Gamma$ and $N_u^{(s)} = \gamma_c/\Gamma$, as discussed in Sec. II B. Therefore, we use the different initial condition $N_c(0) \ll 1$ for our numerical simulations [52]. We keep the condition $N_u(0) = 0$.

For our numerical calculations we take the parameter $U_x/(\hbar\omega a) \cong 0.008$, where $a \equiv \sqrt{\hbar/(m\omega)}$ is the width of the ground state of the harmonic oscillator. As in conventional laser theory [28,29], where one usually has a high-Q cavity, we require $\gamma_c \ll \omega$. Since in the steady state we do not want to have too many atoms in the non-condensate part, we require $\gamma_c \lesssim \Gamma$, as suggested by Eq. (9). In order to have a reasonably small threshold value we choose γ_u appropriately, as indicated by Eq. (12). We also need to choose R_u such that we have a sufficiently large number of atoms in the condensate, that is, $N_c^{(s)} = R_u/\gamma_c - \gamma_u/\Gamma \gg 1$.

2. Average properties of the matter field

In Fig. 1(a) we show the time dependent solutions $N_c(t)$ and $N_u(t)$ of the rate equations (6) and (7). For very small times the atoms accumulate in the un-condensed phase: Due to the small number of atoms in the condensate the Bose enhancement is not yet effective and they cannot make a transition into the condensate. As soon as we have a significant number of atoms in the condensate, N_u rapidly approaches its stationary value $N_u^{(s)} = \gamma_c/\Gamma$. Additional atoms then essentially end up in the condensate where the number of atoms slowly approaches its stationary value $N_c^{(s)} = R_u/\gamma_c - \gamma_u/\Gamma$.

For the numerical analysis of the Eqs. (17) and (18) defining our generic model of the atom laser, we use the same parameters. For the initial condition $\psi(x, t=0)$ of the generalized GPE we use the quantum mechanical ground state of the trap, normalized in such a way that $N_c(0) = \int |\psi(x, t=0)|^2 dx$. In order to get some feeling for the accuracy of our split-operator technique for the generalized GPE we calculate $N_c(t) = \int |\psi(x, t)|^2 dx$ which according to Sec. II A has to coincide with the solution of the rate equations.

We gain insight into the time dependence of $|\psi(x, t)|^2$ by calculating its first moment

$$\overline{x(t)} = \frac{\int x |\psi(x, t)|^2 dx}{\int |\psi(x, t)|^2 dx} \quad (19)$$

and second moment [53]

$$\overline{x^2(t)} = \frac{\int x^2 |\psi(x, t)|^2 dx}{\int |\psi(x, t)|^2 dx}. \quad (20)$$

We note that the symmetry of the trap and the generalized GPE ensure that the symmetry of our initial condition is preserved, that is

$$\overline{x(t)} = 0. \quad (21)$$

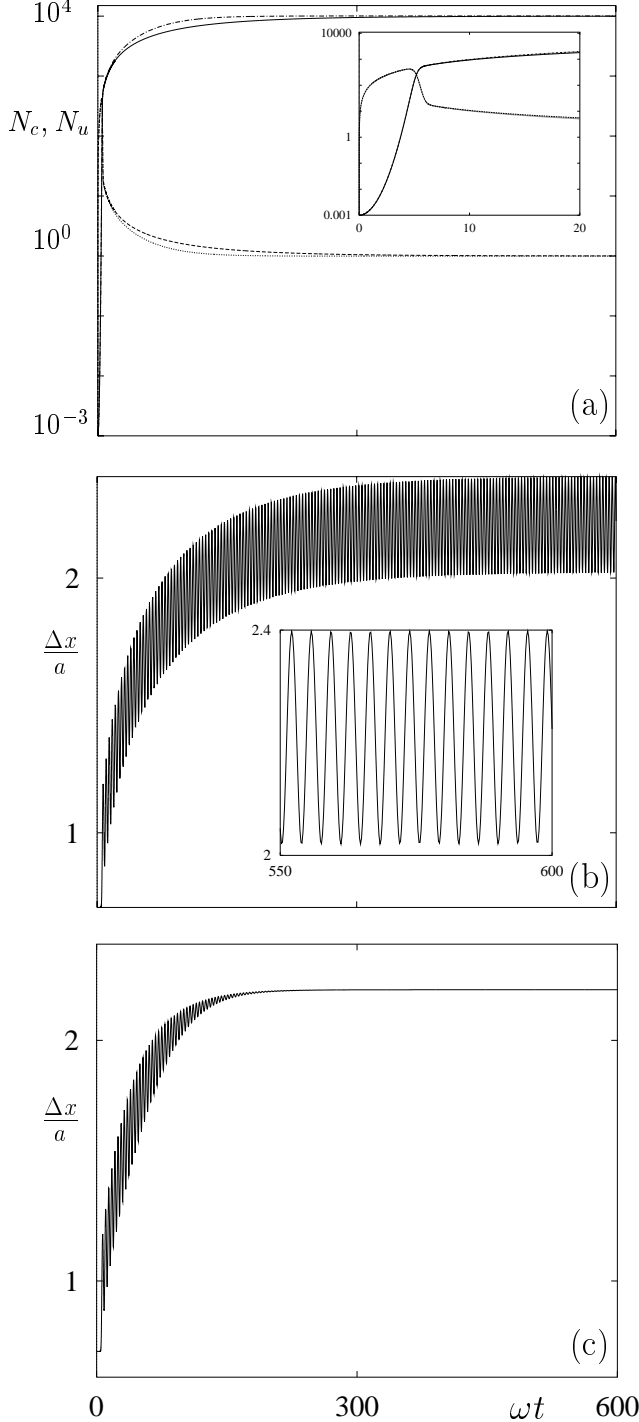


FIG. 1. Generic model of an atom laser: transient regime. (a) The number N_c of atoms in the condensate for the spatially independent (solid line) and spatially dependent (dash-dotted line) loss, and the number of un-condensed atoms N_u for the spatially independent (dashed line) and spatially dependent (dotted line) loss approach their steady-state values 10^4 and 1, respectively. The scale on the vertical axis is logarithmic. The inset shows the early stage of the transient regime. Here, we cannot distinguish the curves corresponding to the different spatial damping, since for small atom numbers the spatial dependence does not play a big role. (b) The width $\Delta x/a$ of the mean field as a function of time displays collective excitations with period $T = 3.57/\omega$ as apparent from the inset. For the spatially homogeneous damping the oscillations are un-damped whereas for the spatially dependent loss they are damped away as shown in (c). The parameters are $R_u/\omega = 10^2$, $\Gamma/\omega = 10^{-2}$, $\gamma_u/\omega = 10^{-2}$, $\gamma_c/\omega = 10^{-2}$, and $U_x/(\hbar\omega a) \cong 0.008$. As initial conditions we have used $N_u(0) = 0$, $N_c(0) = 10^{-3}$, and for $\psi(x, t = 0)$ we have used the quantum mechanical ground state of the trap, normalized in such a way that $\int |\psi(x, t = 0)|^2 dx = N_c(0)$. For (c) we have used a spatially dependent loss $\gamma_c(x)$ described by Eq. (45) with $\gamma'_c/\omega = 0.192$, $x_0/a = 5$, and $\sigma/a = 1$.

Hence the width

$$\Delta x(t) \equiv \sqrt{x^2 - \bar{x}^2} = \sqrt{x^2(t)} \quad (22)$$

of $|\psi(x, t)|^2$ is governed by the second moment, only.

In Fig. 1(b) we plot the width Δx . We note that Δx increases as a function of time and approaches a steady-state value for large times. Moreover, it oscillates around this steady-state value. In the inset we magnify these oscillations. We note that the oscillations do not decay [54]. Therefore Δx does not approach a time independent value and there is no steady state in a strict sense. We hence refer to this solution as the quasi-stationary solution.

In the Appendix we derive an analytical expression for the frequency Ω_2 of these oscillations and find $\Omega_2 = \sqrt{3}\omega \cong 1.73\omega$. From Fig. 1(b) we read off $\Omega_2 \cong 1.76\omega$ which is in good agreement with our prediction.

3. Matter field

The oscillations in the width of the distribution are a manifestation of collective excitations [47] of the condensate. Indeed, the whole distribution oscillates as shown in Fig. 2. Here we display the mean field for a time interval where the numbers of atoms in the condensed state and the un-condensed state have already reached their steady-state value.

In order to study this oscillatory behavior in more detail, we compare in Fig. 3(a) the solution of the time independent GPE (15) in one dimension to the quasi-stationary solution of Eqs. (17) and (18). Here we have depicted $|\psi(x, t)|^2$ for time moments where the width is

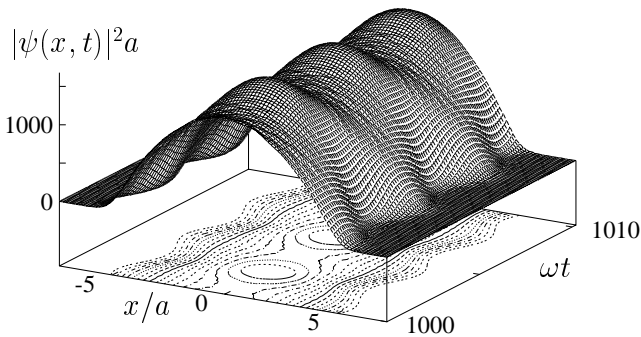


FIG. 2. Quasi-stationary solution $|\psi(x, t)|^2$ of our atom-laser equations for a time interval when the number of atoms in the condensate has already reached its steady-state value. We have used the same parameters as in Fig. 1.

maximal, average, and minimal. Indeed, the solution of Eqs. (17) and (18) oscillates around the solution of the time independent GPE [55–57].

III. IMPROVED MODEL

The collective excitations discussed in the preceding section depend on the initial condition $\psi(x, t = 0)$. This is due to the fact that so far there is no mode selection mechanism in our model which would favor one stationary solution of the GPE over the others. This fact is similar to the optical multi-mode laser [58,59]. We can accomplish a single-mode atom laser when we allow for a space dependent loss. This is the topic of the present section where we formulate an improved model for an atom laser.

A. Formulation of the problem

Real losses in a trapped condensate are spatially dependent: For example, a possible loss mechanism are collisions with un-condensed atoms which are preferably located at the edge of the condensate, see Refs. [1–12]. Further, atoms can be lost at the edge of the condensate because of a finite trapping potential. Moreover, the goal of a matter-wave output-coupler is to create a directed coherent atomic beam. All of these reasons require a spatially dependent loss term. In contrast to this loss term we keep the pump term spatially independent. This reflects the fact that we assume that the condensate is pumped from a cloud of cold thermal atoms which is larger in size.

We therefore consider a modified generalized GPE

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\mathbf{r})\psi + U_0|\psi|^2\psi - \frac{i}{2}\hbar\gamma_c(\mathbf{r})\psi + \frac{i}{2}\hbar\Gamma N_u\psi, \quad (23)$$

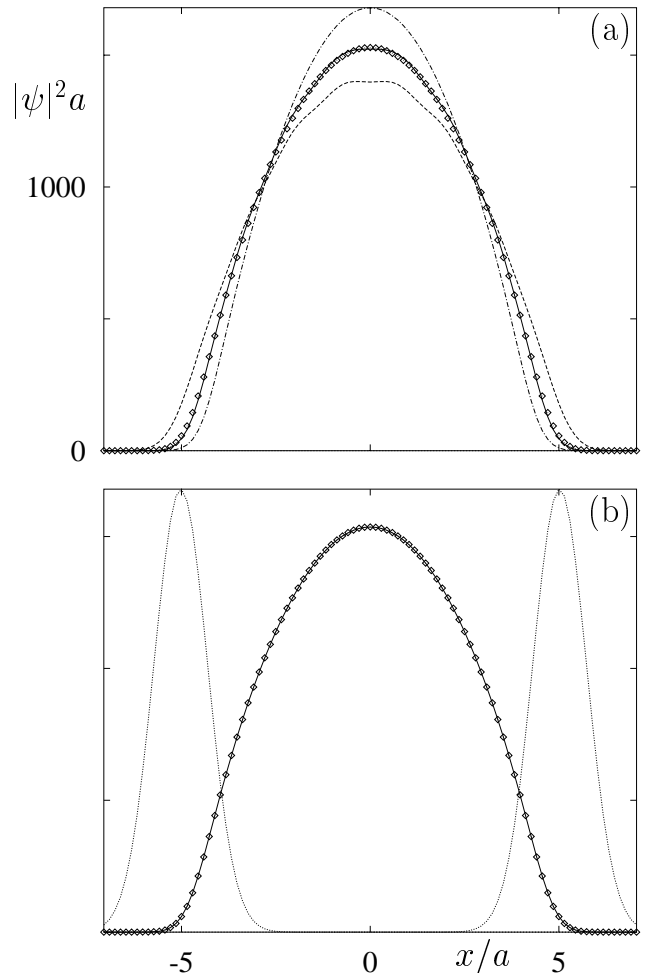


FIG. 3. Quasi-stationary and stationary solutions of our atom-laser equations. On the top (a) we compare the solution of the time independent Gross-Pitaevskii equation (diamonds) to the quasi-stationary solution of Eq. (17) for times where the width is maximal (dashed line), average (solid line), and minimal (dash-dotted line), respectively. These times correspond to $\omega t = 1000.54, 1001.45, 1002.35$. On the bottom (b) we consider the case of spatially dependent loss. We depict (not to scale) the loss rate $\gamma_c(x)$ by a dotted line. In this case collective excitations are damped out and in steady-state the solutions of our atom laser equations (23) and (25) coincide with the time independent ground-state solution (diamonds) of Eqs. (28) and (29). In both figures we have used the parameters of Fig. 1.

where the decay rate $\gamma_c(\mathbf{r})$ is spatially dependent. Hence, the rate equation (7) for the number N_c of condensed atoms now translates into

$$\dot{N}_c = \Gamma N_u N_c - \int \gamma_c(\mathbf{r}) |\psi(\mathbf{r}, t)|^2 d^3r. \quad (24)$$

Note that due to the space dependent loss rate we cannot express the integral in Eq. (24) by N_c . Therefore we do not have a rate equation for N_c anymore. However, the rate equation

$$\dot{N}_u = R_u - \gamma_u N_u - \Gamma N_c N_u \quad (25)$$

for the number N_u of un-condensed atoms remains the same.

Equations (23) and (25) are the two fundamental equations of our improved model of an atom laser.

B. Stationary solution

We now study the stationary solution of Eqs. (23) and (25). We first derive the steady-state expressions and then discuss the laser threshold in the presence of a position dependent loss term. We conclude by introducing a modified Thomas-Fermi approximation.

1. Matter field in steady-state

When we follow an analysis similar to the one of Sec. II B, we find two sets of stationary solutions of the Eqs. (23), (24), and (25). The first set with

$$\begin{aligned} N_u^{(s)} &= \frac{R_u}{\gamma_u}, \\ N_c^{(s)} &= 0, \\ \psi^{(s)} &= 0 \end{aligned} \quad (26)$$

corresponds to the solution below threshold.

The second set

$$N_u^{(s)} = \frac{1}{\Gamma} \int \gamma_c(\mathbf{r}) |\psi_1^{(s)}(\mathbf{r})|^2 d^3r = \frac{R_u}{\gamma_u + \Gamma N_c^{(s)}}, \quad (27)$$

$$N_c^{(s)} = \frac{R_u}{\int \gamma_c(\mathbf{r}) |\psi_1^{(s)}(\mathbf{r})|^2 d^3r} - \frac{\gamma_u}{\Gamma} \quad (28)$$

corresponds to the solutions above threshold as we show now. Here, $\psi_1^{(s)}(\mathbf{r}) = [N_c^{(s)}]^{-1/2} \psi^{(s)}(\mathbf{r})$ denotes the stationary solution of Eq. (23) normalized to unity. This solution is defined by the time-independent modified generalized GPE

$$\begin{aligned} \mu \psi^{(s)} &= -\frac{\hbar^2}{2m} \Delta \psi^{(s)} + V(\mathbf{r}) \psi^{(s)} + U_0 |\psi^{(s)}|^2 \psi^{(s)} \\ &\quad - \frac{i}{2} \hbar \gamma_c(\mathbf{r}) \psi^{(s)} + \frac{i}{2} \hbar \Gamma N_u \psi^{(s)}, \end{aligned} \quad (29)$$

following from Eq. (23) using the ansatz $\psi(\mathbf{r}, t) = \exp(-i\mu t/\hbar) \psi^{(s)}(\mathbf{r})$ where μ denotes the “chemical potential.”

These expressions for $N_u^{(s)}$ and $N_c^{(s)}$ are not explicit since they involve the stationary solution $\psi^{(s)}(\mathbf{r})$ of Eq. (29). In Sec. III B 3 we derive an approximate analytical expression and in Sec. III B 5 we find a fully numerical solution for $\psi^{(s)}(\mathbf{r})$.

2. Lasing threshold

A stability analysis of the above solutions amounts to calculating the collective excitations of the modified generalized GPE (23). This is only possible numerically. We therefore pursue a strategy where we solve numerically for the full time dependence of $\psi(\mathbf{r}, t)$. In order to gain some insight into the threshold condition we first discuss simple physical arguments.

These considerations rely on the fact that the number of atoms cannot be negative. With the help of Eq. (28) we then determine the laser threshold

$$R^{th} \equiv \frac{\gamma_u}{\Gamma} \int \gamma_c(\mathbf{r}) |\psi_1^{(s)}(\mathbf{r})|^2 d^3r \quad (30)$$

by setting $N_c^{(s)} = 0$. Close to threshold the number $N_c^{(s)}$ of atoms in the condensate is close to zero. We can therefore neglect the non-linear contribution in the modified generalized GPE and arrive at a linear Schrödinger equation with pump and loss. With a position dependence of the loss term that favors the normalized ground-state energy solution $\phi_0(\mathbf{r})$ of the linear Schrödinger equation, we can replace $\psi_1^{(s)}(\mathbf{r})$ by $\phi_0(\mathbf{r})$ in Eq. (30) and arrive at

$$R^{th} = \frac{\gamma_u}{\Gamma} \int \gamma_c(\mathbf{r}) |\phi_0(\mathbf{r})|^2 d^3r. \quad (31)$$

When the loss shape is flat around the localization of $\phi_0(\mathbf{r})$, that is, around the center $\mathbf{r} = \mathbf{0}$ of the trap, we can factor out $\gamma_c(\mathbf{0})$ from the integral in Eq. (31) and find the lasing threshold

$$R^{th} = \frac{\gamma_u \gamma_c(\mathbf{0})}{\Gamma}. \quad (32)$$

We conclude this section by noting that in the limit of a space independent loss the above results reduce to the corresponding ones of Sec. II B 1.

3. Modified Thomas-Fermi solution

In this section we derive an approximate but analytical expression for the stationary state of the modified generalized GPE. In the case of the conventional GPE it is the so-called Thomas-Fermi (TF) approximation which describes the steady state of the condensate [60]. The phase of the stationary solution of the GPE as well as the phase of the TF approximation is constant. However, for a position dependent loss it turns out that the phase of the stationary matter-wave field also depends on the position. A spatially dependent phase leads to a non-vanishing current of the condensate. We expect this feature to be important for the coherence properties of a cw atom laser. Therefore, we introduce a modified TF solution.

We start from the time independent form of the modified generalized GPE, Eq. (29). For the present problem

a hydrodynamic treatment is more convenient. We therefore consider the density

$$\rho(\mathbf{r}) \equiv |\psi^{(s)}(\mathbf{r})|^2 \quad (33)$$

and velocity

$$\mathbf{v}(\mathbf{r}) \equiv (\hbar/m)\nabla\phi(\mathbf{r}) \quad (34)$$

of the condensate. Here $\phi(\mathbf{r})$ is the phase of the mean field following from the ansatz

$$\psi^{(s)}(\mathbf{r}) = \sqrt{\rho(\mathbf{r})}e^{i\phi(\mathbf{r})}. \quad (35)$$

Indeed, with Eq. (29) we find the equations

$$\frac{\hbar^2}{2m}\nabla^2\sqrt{\rho} = \left[-\mu + \frac{1}{2}m\mathbf{v}^2 + V(\mathbf{r}) + U_0\rho\right]\sqrt{\rho} \quad (36)$$

and

$$\frac{\mathbf{v}}{2}\nabla\sqrt{\rho} = [-\nabla\cdot\mathbf{v} - \gamma_c(\mathbf{r}) + \Gamma N_u]\sqrt{\rho}. \quad (37)$$

We make the assumption that the density profile is slowly varying. Since we are interested in loss shapes where the loss at the center of the condensate is uniform, these assumptions are reasonable at the center of the condensate. However, at the edges of the condensate where the change of the loss is large, we expect this assumption to break down. This is the idea of our modified Thomas-Fermi approximation.

This assumption allows us to neglect the terms $\nabla^2\sqrt{\rho}$ and $\nabla\sqrt{\rho}$ in the two equations. However, in order to fulfill the second equation we have to retain the derivative $\nabla\cdot\mathbf{v}$ in velocity. In this approximation we arrive at the approximate expression

$$\rho(\mathbf{r}) \cong \frac{1}{U_0} \left[\mu - \frac{1}{2}m\mathbf{v}^2(\mathbf{r}) - V(\mathbf{r}) \right] \quad (38)$$

for the density and the differential equation

$$\nabla\cdot\mathbf{v}(\mathbf{r}) \cong -\gamma_c(\mathbf{r}) + \Gamma N_u \quad (39)$$

for the velocity.

Since the density cannot be negative, these expressions are only valid for that volume \mathcal{V} of space where

$$\rho(\mathbf{r}) \propto \mu - \frac{1}{2}m\mathbf{v}^2(\mathbf{r}) - V(\mathbf{r}) \geq 0. \quad (40)$$

The shape of this volume \mathcal{V} depends on the potential $V(\mathbf{r})$ and the loss rate $\gamma_c(\mathbf{r})$ via the velocity $\mathbf{v}(\mathbf{r})$.

Note, that the velocity \mathbf{v} and consequently the phase ϕ depends on the rate Γ , the number of un-condensed atoms, and the shape of the loss rate. Indeed, it is only the spatial dependence of the loss rate that determines the spatial profile of the velocity $\mathbf{v}(\mathbf{r})$. The “kinetic energy” of the condensate plays a role similar to the trap potential in shaping the density.

The expressions, Eqs. (38) and (39), for the density and velocity are not explicit. Indeed, the chemical potential μ is still a free parameter. Moreover, the number of un-condensed atoms $N_u^{(s)}$ is coupled to Eqs. (38) and (39) and is given by Eq. (27).

We therefore have to solve the various constraints in a self-consistent way: The number $N_c^{(s)}$ of condensed atoms reads

$$N_c^{(s)} = \int_{\mathcal{V}} \rho(\mathbf{r}) d^3r. \quad (41)$$

On the other hand this quantity $N_c^{(s)}$ is given by Eq. (28) and reads

$$N_c^{(s)} = \frac{R_u}{\int_{\mathcal{V}} \gamma_c(\mathbf{r})\rho_1(\mathbf{r}) d^3r} - \frac{\gamma_u}{\Gamma}, \quad (42)$$

where $\rho_1(\mathbf{r}) = [N_c^{(s)}]^{-1}\rho(\mathbf{r})$ is the density normalized to unity. Note, that \mathcal{V} is defined by the condition that the density is non-negative, Eq. (40).

The number of un-condensed atoms then follows from Eq. (27). Hence we have to solve the three Eqs. (40), (41), and (42) for the three unknowns: the modified Thomas-Fermi volume \mathcal{V} , the chemical potential μ , and the number $N_c^{(s)}$ of atoms in the condensate.

4. Usual Thomas-Fermi solution

The most important consequence of our position dependent loss term is a non-vanishing velocity and therefore a spatially dependent phase. How does this compare to the usual Thomas-Fermi approximation? Here we consider the density

$$\rho(\mathbf{r}) \cong \frac{\mu - V(\mathbf{r})}{U_0} \quad (43)$$

and the velocity

$$\mathbf{v} \equiv 0. \quad (44)$$

We first note that this ansatz is in contradiction to Eq. (37). Nevertheless we can use it to investigate how the kinetic energy potential influences the density ρ . Similar to the last section we insert our ansatz into Eqs. (40) to (42) and find the Thomas-Fermi volume \mathcal{V} , the chemical potential μ , and the number of atoms $N_c^{(s)}$.

5. Exact solution

In this section we adapt the numerical methods developed to find the ground-state solution of the ordinary GPE [55–57] to the present problem. This method evolves the wave function for a fixed atom number in imaginary time. The solution is normalized to unity after each time step. The evolution in imaginary time attenuates the differences between the arbitrary initial wave function and the ground-state solution of the ordinary GPE.

In our model of an atom laser, Eq. (23), we have generalized the GPE by a pump and a loss term. Therefore the number of atoms is not fixed but is governed by Eq. (25). In order to find the ground-state solution of our time independent Eqs. (28) and (29), we evolve an arbitrary initial wave function according to our modified generalized GPE, Eq. (23), in imaginary time, using the split-operator technique [48,49]. After each time step we normalize the wave function to unity. Then we update the number of atoms with the help of Eq. (28) and use it for the next time step. We repeat this procedure until the wave function and the atom number has converged to a stationary value. This method finds the stationary state of our improved model of an atom laser in a self-consistent way. The most important result is that the pump and spatial loss gives a space dependent phase to the stationary mean-field $\psi^{(s)}(\mathbf{r})$.

C. Results

As in Sec. II we now specify the potential $V(\mathbf{r})$ to be the one-dimensional harmonic oscillator potential, Eq. (16). Moreover, we study two models of a position dependent loss rate: a sum

$$\gamma_c(x) = \gamma'_c \left(e^{-(x+x_0)^2/\sigma^2} + e^{-(x-x_0)^2/\sigma^2} \right) \quad (45)$$

of two Gaussians and a sum

$$\gamma_c(x) = \gamma'_c \left[\frac{\sigma^2}{(x+x_0)^2 + \sigma^2} + \frac{\sigma^2}{(x-x_0)^2 + \sigma^2} \right]. \quad (46)$$

of two Lorentzians.

In both cases σ is a measure of the width of the corresponding distributions and γ'_c is the maximal loss rate when the two curves have negligible overlap. In the next section we choose the locations $\pm x_0$ of the maximal loss such that they sit where the density of the condensate falls off.

1. Transient behavior

First we want to show that the collective excitations which emerged in the elementary model are damped out

due to the presence of the space dependence in the loss [61]. For this analysis we use the Gaussian loss term, Eq. (45). We adjust γ'_c such that we obtain the same final number of atoms in the condensate as in the spatially independent case. The width of the loss is comparable to a typical scale, such as the width of the ground state of the harmonic oscillator.

In Fig. 1(a) we compare the time evolution of the numbers of condensed and un-condensed atoms for the position dependent loss with those of the position independent loss. We see that the overall behavior is not that different. Therefore our original rate equations of the elementary model still approximate very well the time evolution of N_c and N_u following from the improved model.

We then show in Fig. 1(c) the scaled width of the mean field. Indeed, the position dependent loss damps out the collective excitations, and a steady state is reached. We emphasize that this is true for any initial non-vanishing mean field.

In Fig. 3(b) we compare and contrast the stationary mean field to the numerical solution of the time independent equation. They coincide with each other. Indeed, for many initial conditions and set of parameters we have noticed that the mean field obtained by evolving the modified generalized GPE over a sufficiently long enough time such that the transient oscillations are damped out is equivalent to the time independent ground-state energy solution of the modified generalized GPE. This holds as long as the loss is confined to the edges of the condensate [62]. This is the case in the present example as shown in Fig. 3(b) where we also display the loss function in arbitrary units. The overlap between the stationary mean field and the loss function is essential for the number of atoms in the condensate as is apparent from Eq. (28).

When we compare Fig. 3(a) and (b), we note that the modulus of the steady-state mean field in the presence of a loss located at the edges of the condensate is very similar to the ground-state energy solution of the conventional time independent GPE, Eq. (15). However, this is not true for the phase of the mean field as already discussed in Sec. IIIB.

A spatially dependent loss term located at the edges of the condensate damps out the collective excitations. In our theory of an atom laser this plays the role of mode selection, i.e., the ground-state energy solution survives whereas the excited states, i.e., the collective excitations, are damped away due to the fact that their overall spread in position space is greater than that of the ground state.

2. Stationary solution

(a) *Modified Thomas-Fermi solution.* We now turn to the discussion of the modified Thomas-Fermi solution derived in Sec. IIIB 3 for a general potential. In the present section we restrict this analysis to a one-dimensional harmonic oscillator potential, Eq. (16). Moreover, we choose

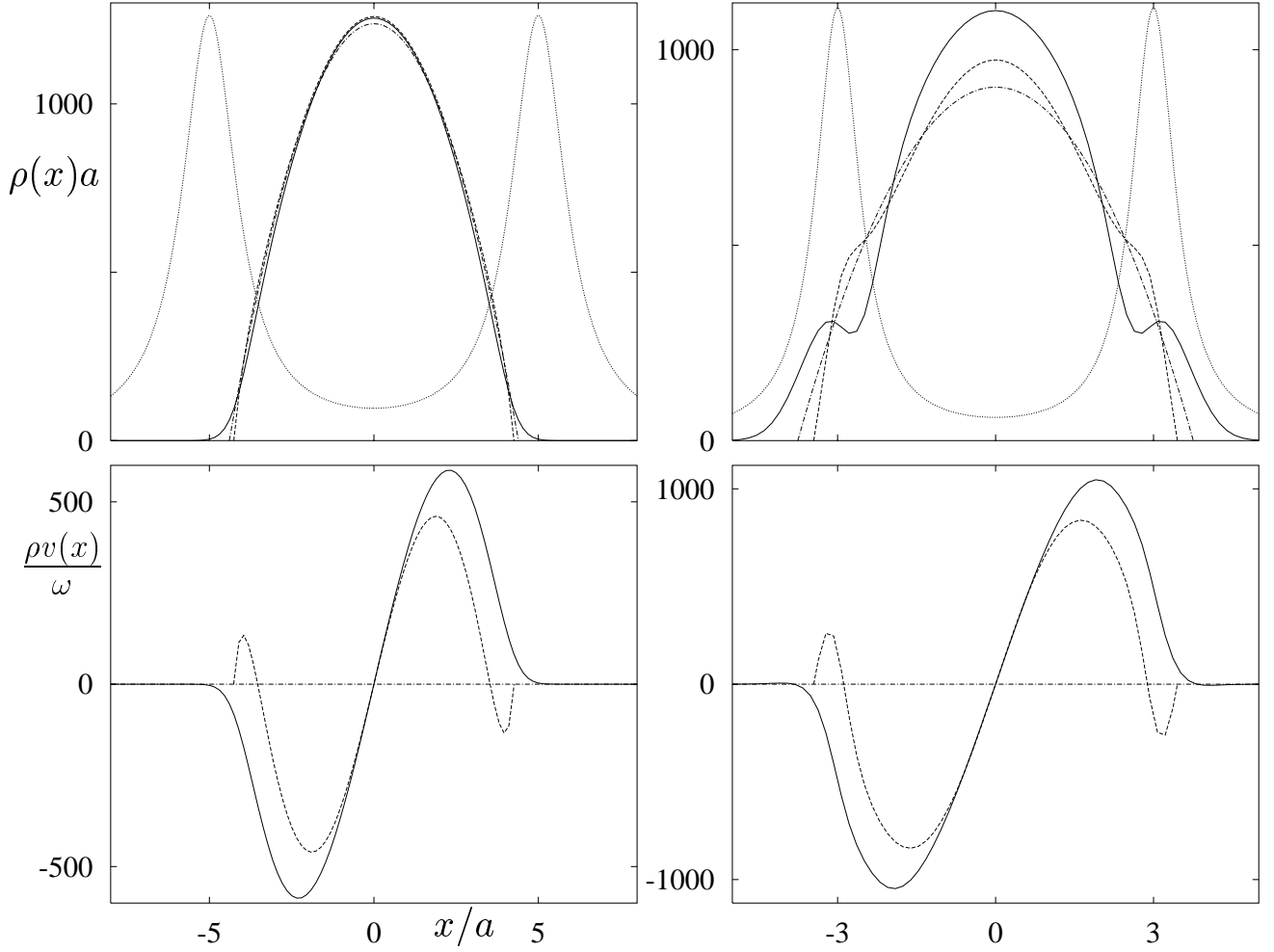


FIG. 4. Comparison between the two Thomas-Fermi and the numerical solutions of the atom laser equations in the presence of space dependent loss. We display density (upper part) and current (lower part) of the numerical (solid line), modified TF (dashed line), and usual TF (dash-dotted line) solution. On the top we also show the space dependence of the loss term $\gamma_c(x)$. Note the position of the peaks, the height is not to scale. Here we have used the parameters $R_u/\omega = 5 \times 10^3$, $\Gamma/\omega = 0.5$, $\gamma_u/\omega = 1$, $\gamma'_c/\omega = 5$, and $U_x/(\hbar\omega a) \cong 0.008$. In the left column the loss is concentrated at $x = \pm 5a$, whereas on the right it is at $x = \pm 3a$. Moreover, the widths are different $\sigma/a = 1$ (left) and $\sigma/a = 0.5$ (right).

the Lorentzian loss rate, Eq. (46). This choice is solely motivated by the fact that we can perform the resulting integral of the differential equation (39).

We start by first summarizing the Eqs. (40) to (42) for the chemical potential μ , the Thomas-Fermi radius R , and the number of condensed atoms $N_g^{(s)}$,

$$\mu - \frac{1}{2}m[v(R)]^2 - \frac{1}{2}m\omega^2 R^2 = 0, \quad (47)$$

$$N_c^{(s)} - \int_{-R}^R \rho(x) dx = 0, \quad (48)$$

$$N_c^{(s)} - \frac{R_u}{\int_{-R}^R \gamma_c(x) \rho_1(x) dx} - \frac{\gamma_u}{\Gamma} = 0, \quad (49)$$

determining the modified Thomas-Fermi solution for the one-dimensional harmonic trap and loss. From Eq. (38) we obtain the one-dimensional density

$$\rho(x) = \frac{1}{U_x} \left[\mu - \frac{1}{2}mv^2(x) - \frac{1}{2}m\omega^2 x^2 \right]. \quad (50)$$

We find the velocity v when we substitute the Lorentzian loss rate, Eq. (46), into Eq. (39) and integrate which yields

$$v(x) = \Gamma N_u x - \gamma'_c \sigma \left[\arctan\left(\frac{x+x_0}{\sigma}\right) + \arctan\left(\frac{x-x_0}{\sigma}\right) \right]. \quad (51)$$

Here we have set $v(0) = 0$ in order to preserve the symmetry of the solution of the mean field, Eq. (50), $\rho(x) = \rho(-x)$. The latter holds because the harmonic oscillator potential, the spatially dependent loss, and the generalized GPE show this symmetry.

(b) *Usual Thomas-Fermi solution.* In order to get a feeling how in our one-dimensional model the density ρ is influenced by the velocity v we also discuss the usual TF solution. For the one-dimensional harmonic trap the density, Eq. (43), reads

$$\rho(x) \cong \frac{1}{U_x} \left[\mu - m \frac{1}{2} \omega^2 x^2 \right], \quad (52)$$

and the velocity, Eq. (44), reads

$$v(x) \equiv 0. \quad (53)$$

When we use this usual TF approximation, we easily find from Eq. (47) with $v \equiv 0$ the TF radius $R = \sqrt{2\mu/m\omega^2}$. We then use Eq. (48) to calculate the number of atoms in the condensate as a function of the chemical potential μ . Inverting this equation we can express the chemical potential $\mu = [3N_c^{(s)} U_x \sqrt{m\omega}/(4\sqrt{2})]^{2/3}$ as a function of the atom number $N_c^{(s)}$. Finally, we use Eq. (49) to determine the number $N_c^{(s)}$ of condensed atoms, the only unknown quantity. We emphasize that only this last step has to be done numerically.

(c) *Discussion.* Figure 4 shows the modified TF, the usual TF and the fully numerical solution. We have chosen two different shapes of the Lorentzian loss curve: In the right column the width of the individual Lorentzians is half the size of the one on the left column. Moreover, we note that their location is different. We depict the modified TF solution by dashed curves, the usual TF solution by dash-dotted ones, and the numerical solution by solid lines. Again we show in arbitrary units the shape of the loss $\gamma_c(x)$ by the dotted curve.

The density is shown in the upper part of Fig. 4. We see that for the parameters of the left column both TF approximations for the density work quite well. The modified TF solution approximates the center better than the usual TF solution. However, the radius derived from the usual TF approximation is a better estimate of the edge of the condensate than the modified TF radius. For the set of parameters used in the right column of Fig. 4 both TF approximations for the density do not work that well anymore. The reason for this break-down is that the condensate reaches too far into the loss region where the density varies strongly and the derivatives neglected in the derivation of Eqs. (38) and (39) become important. Nevertheless our modified TF solution shows at least qualitatively the same behavior as the fully numerical solution. Surprisingly, at regions towards the peaks of the loss where the loss rate is at its highest, the density is first decreased and then increased in the modified TF over the usual TF solution. This is understandable from Eq. (38) when we consider in the lower part of Fig. 4 the current $j(x) = \rho(x)v(x)$. We note that at these regions first the current which is proportional to the velocity is at its highest and then decreases.

We see that for the current, i.e. the velocity, the modified TF solution is a very good approximation around

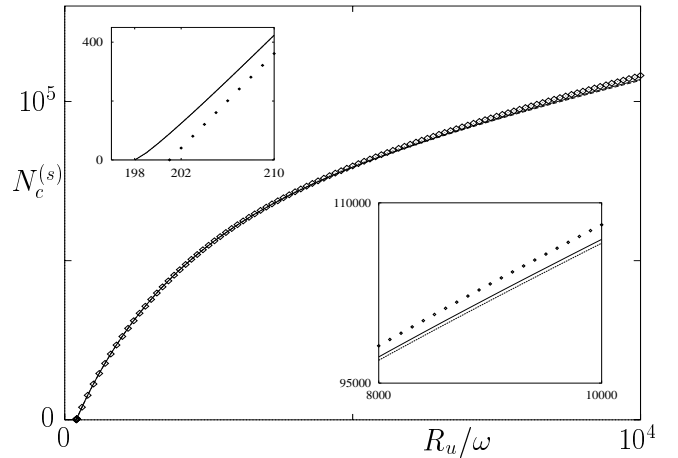


FIG. 5. Comparison between analytical, but approximate, and numerical, but exact solutions of the atom laser equations. For this comparison we analyze the threshold behavior of the atom laser: Condensate population $N_c^{(s)}$ in steady-state as a function of the pump strength R_u . Over a wide range of pump strengths both approximate solutions, the modified (solid line) and the usual (dashed line) TF solution accurately describe the fully numerical solution (diamonds). In the region close to threshold the two approximate solutions, which are almost identical, clearly deviate from the fully numerical solution as shown in the inset in the upper left corner. The other inset magnifies a region far above threshold where the modified TF solution is a better approximation than the usual TF solution. The parameters used are: $\Gamma/\omega = 10^{-3}$, $\gamma_u/\omega = 10$, $\gamma_c'/\omega = 1$, $\sigma/a = 1$ and $x_0/a = 10$, and $U_x/(\hbar\omega a) \cong 0.008$.

the central region. It gives a good estimate for the overall behavior of the current (velocity) for both sets of parameters. Around the central region of the condensate the current is approximately a linear function of the position, illustrating the flow of atoms to the ends where they are predominately lost from the peaks. At the tails of the condensate the modified TF solution even predicts a velocity changing the direction. This is in contrast to the fully numerical solution. The usual TF solution has zero velocity by default.

We solve numerically the Eqs. (47) to (49) for the modified Thomas-Fermi solution to find the condensate population for various pump strengths and display this as the solid curve in Fig. 5. The dashed curve corresponds to the usual TF solution. We display the fully numerical solution with the help of diamonds.

Both approximate analytical curves agree quite well with the fully numerical solution over a wide region of different pump strengths. However, at pump rates just above threshold the agreement is not that good which is shown in the upper left inset of Fig. 5. Both the usual TF and the modified TF curves lie on top of each other and over-estimate the numerical result. The fully numerical solution crosses the horizontal axis at $R_u/\omega \cong 201$ whereas the two TF solutions cross at $R_u/\omega \cong 198$. Close

to threshold, the spatial extent of the two approximate wave functions is becoming smaller with decreasing atom number. They are predominately located around the relatively flat loss region and the losses appear to be position independent. Therefore the laser threshold for the two approximate solutions is predicted by Eq. (32). In contrast, the shape of the numerically calculated mean field can vary, and is in fact of Gaussian form close to threshold. Hence our approximation of the threshold, Eq. (31), agrees rather remarkably with the numerically calculated threshold. The deviation of the approximate threshold, Eq. (32), reflects the fact that even close to threshold the loss function does not appear spatially constant to the mean field for the parameters of Fig. 5. When we recall that the TF approximation is not good at low atom numbers it is rather surprising to see such small differences between the TF solutions and the numerical solution.

The lower right inset of Fig. 5 zooms in on a region of higher pump strengths far above threshold where the spatial structure of the loss plays a role. Here the modified TF solution approximates the fully numerical solution (diamonds) better than the usual TF solution. This is understandable since the modified TF solution takes into account pump and spatially dependent loss by allowing for a spatially dependent phase, i.e. velocity.

IV. CONCLUSIONS

In summary, we have constructed a theory of an atom laser that is analogous to semi-classical laser theory. The matter-wave equation is a generalized Gross-Pitaevskii equation with additional loss and gain terms. We derive the lasing threshold and describe the build-up of the coherent mean field of a condensate.

The elementary model uses a spatially homogeneous loss. Here we find un-damped collective excitations. Therefore the final mean field depends on the initial mean field: The known stationary ground state of the GPE which is the desired lasing mode cannot be reached in general.

The improved model has a natural mode selection built in by a space dependent loss. In this way we achieve the desired single lasing mode.

We have derived a modified Thomas-Fermi solution for the steady-state mean field. This solution takes into account the effects of a pump term and a position dependent loss term. In contrast to a constant phase of the usual Thomas-Fermi solution, the modified Thomas-Fermi solution has a spatially dependent phase, i.e. velocity, due to the permanent flow of atoms in and out of the condensate. The modified Thomas-Fermi solution is a good approximation in regions where the loss shape is slowly varying and for sufficiently large atom numbers.

We emphasize that our model of an atom laser is very simple and rather general. Therefore, we can apply it to different experimental configurations of cw atom lasers

[63], or current experiments [1–12] provided the evaporative cooling process (boson amplification) and the loading of the trap is run continuously, and an output coupling mechanism is applied.

ACKNOWLEDGMENTS

We thank Eric Bolda, Michael Fleischhauer, Murray Olsen, Karl Riedel, and Janne Ruostekoski for stimulating and valuable discussions. Two of us (B.K. and K.V.) acknowledge very gratefully the warm hospitality given to them at their stay with the quantum optics group of the department of physics at the University of Auckland. This work was supported by the Deutsche Forschungsgemeinschaft, the University of Auckland Research Committee and the Marsden Fund of the Royal Society of New Zealand.

APPENDIX: COLLECTIVE EXCITATIONS IN ONE DIMENSION

Collective excitations are usually discussed in three dimensions [47]. Within the framework of the Thomas-Fermi approximation, Stringari [51] has calculated analytically the excitation spectrum of a condensate in a three-dimensional isotropic harmonic trap. Since our numerical solution of Eq. (5) is done for a one-dimensional harmonic trap, we apply Stringari's method to a one-dimensional harmonic trap of frequency ω . We substitute

$$\psi(x, t) = \sqrt{\rho(x, t)} e^{i\phi(x, t)} \quad (\text{A1})$$

into Eq. (5) and obtain after some algebra the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) + (\gamma_c - \Gamma N_u) \rho = 0 \quad (\text{A2})$$

and

$$m \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left(\frac{m}{2} v^2 + V + U_x \rho - \mu \right) - \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x^2} \sqrt{\rho} = 0 \quad (\text{A3})$$

for the “density”

$$\rho(x, t) = \psi^*(x, t) \psi(x, t) \quad (\text{A4})$$

and the “velocity”

$$v(x, t) = \frac{\hbar}{m} \frac{\partial \phi(x, t)}{\partial x}. \quad (\text{A5})$$

Note, that we have also introduced the chemical potential μ which is space independent.

We now consider small deviations

$$\begin{aligned}
\delta\rho &\equiv \rho - \rho^{(s)}, \\
\delta v &\equiv v - v^{(s)} = v, \\
n_u &\equiv N_u - N_u^{(s)} = N_u - \gamma_c/\Gamma
\end{aligned} \tag{A6}$$

of ρ , v , and N_u from their stationary values $\rho^{(s)}$, $v^{(s)} \equiv 0$, and $N_u^{(s)} = \gamma_c/\Gamma$. Furthermore, we make the Thomas-Fermi approximation, that is, we neglect the last term in Eq. (A3) and approximate the stationary solution $\rho^{(s)}(x)$ by the Thomas-Fermi solution [60]

$$\rho^{(s)}(x) \cong \frac{\mu - V(x)}{U_x}, \tag{A7}$$

where the chemical potential μ is defined by the normalization integral. We then arrive at the linearized equations

$$\frac{\partial}{\partial t} \delta\rho + \frac{\partial}{\partial x} (\rho^{(s)} v) - \Gamma \rho^{(s)} n_u = 0 \tag{A8}$$

and

$$m \frac{\partial v}{\partial t} + U_x \frac{\partial}{\partial x} \delta\rho = 0. \tag{A9}$$

We combine these two equations to eliminate v and arrive at

$$\frac{\partial^2}{\partial t^2} \delta\rho - \frac{U_x}{m} \frac{\partial}{\partial x} \left(\rho^{(s)} \frac{\partial}{\partial x} \delta\rho \right) = \Gamma \rho^{(s)} \dot{n}_u. \tag{A10}$$

For large times, when $N_u(t)$ and $N_c(t)$ have already reached their stationary value, we can neglect the inhomogeneous term $\Gamma \rho^{(s)} \dot{n}_u$.

Equation (A10) cannot be solved without knowledge of the potential $V(x)$. We restrict ourselves to the case of a harmonic trap, that is,

$$V(x) = \frac{1}{2} m \omega^2 x^2. \tag{A11}$$

In order to solve Eq. (A10), we introduce the scaled variable $\xi = x/R$, where $R = \sqrt{2\mu/m\omega^2}$ is the Thomas-Fermi radius of the condensate. Using the ansatz

$$\delta\rho(x, t) = A \sin(\Omega t + \varphi) y(x/R) \tag{A12}$$

we obtain an ordinary differential equation for $y(\xi)$ which reads

$$\frac{d}{d\xi} \left[(1 - \xi^2) \frac{dy(\xi)}{d\xi} \right] + \frac{2\Omega^2}{\omega^2} y(\xi) = 0. \tag{A13}$$

This is the differential equation of Legendre functions which in general only has solutions that are singular at $\xi = \pm 1$ [64], that is at $x = \pm R$. The only exceptions are

$$\frac{2\Omega_n^2}{\omega^2} = n(n+1), \tag{A14}$$

where n is an integer. In this case the well-known Legendre polynomials [64]

$$P_n(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n \tag{A15}$$

solve Eq. (A13). Furthermore, they fulfill the orthogonality relation

$$\int_{-1}^{+1} P_n(\xi) P_m(\xi) d\xi = 0 \quad \text{for } n \neq m. \tag{A16}$$

The frequencies of the elementary excitations are therefore given by Eq. (A14). The solution of the homogeneous part of Eq. (A10) reads

$$\delta\rho(x, t) = \sum_{n=1}^{\infty} A_n \sin(\Omega_n t + \varphi_n) P_n(x/R), \tag{A17}$$

where A_n and φ_n follow from the initial deviation from the stationary solution.

Equation (A17) shows that the excitations do not decay, even in the presence of pump and loss terms in our generalized Gross-Pitaevskii equation, Eq. (17). However, these excitations do not grow but rather oscillate. Our numerical solution of Eq. (17) in Sec. II C confirms this result.

In Fig. 1(b) we do not find all frequencies allowed by Eq. (A14) since in our numerical solution of Eqs. (17) and (18) we start with a symmetric initial condition $\psi(x, t = 0)$. Because Eqs. (17) and (18) do not destroy this symmetry, $\rho(x, t)$ as well as $\delta\rho(x, t)$, the deviation from the (symmetric) stationary solution of the generalized GPE, will always be symmetric. We therefore can only find excitations which correspond to Legendre polynomials of even order if we start with a symmetric $\psi(x, t = 0)$. The corresponding frequencies are

$$\Omega_{2n} = \omega \sqrt{n(2n+1)}. \tag{A18}$$

This is also true for an anti-symmetric initial wave function since the corresponding density is symmetric.

Two other facts are worth mentioning: The excitations discussed above do not change the number of atoms. Using $P_0(\xi) = 1$ we find with the help of the orthogonality relation Eq. (A16)

$$\int \delta\rho(x, t) dx = 0. \tag{A19}$$

The second moment

$$\overline{x^2(t)} = \frac{\int x^2 |\psi(x, t)|^2 dx}{\int |\psi(x, t)|^2 dx} \tag{A20}$$

can only oscillate with the frequency $\Omega_2 = \sqrt{3}\omega$. This follows from the relation

$$\xi^2 = \frac{2}{3} P_2(\xi) + \frac{1}{3} P_0(\xi) \tag{A21}$$

together with the orthogonality relation, Eq. (A16). A generalization is the statement that

$$\overline{P_k(x/R)} = \frac{\int P_k(x/R) |\psi(x,t)|^2 dx}{\int |\psi(x,t)|^2 dx} \quad (\text{A22})$$

only shows oscillations with frequency Ω_k .

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